Group actions on simple stably finite C^*-algebras

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Notations

- A group Γ is always assumed to be countable, discrete and amenable.
- A C^*-algebra A is always assumed to be unital, simple and separable.
- $U(A)$ denotes the group of unitaries of A, and $\mathrm{Aut}(A)$ denotes the group of automorphisms of A.
- For $u \in U(A)$, $\mathrm{Ad}\, u \in \mathrm{Aut}(A)$ is given by $x \mapsto uxu^*$ and is called an inner automorphism.
- $\mathcal{Z}$ denotes the Jiang-Su algebra.

Cocycle actions

Definition

A pair (α, u) of a map $\alpha : \Gamma \to \mathrm{Aut}(A)$ and a map $u : \Gamma \times \Gamma \to U(A)$ is called a cocycle action of Γ on A if

$$\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g, h) \circ \alpha_{gh}$$

and

$$u(g, h)u(gh, k) = \alpha_g(u(h, k))u(g, hk)$$

hold for any $g, h, k \in \Gamma$. We write $(\alpha, u) : \Gamma \curvearrowright A$.

We always assume $\alpha_1 = \mathrm{id}$, $u(g, 1) = u(1, g) = 1$ for all $g \in \Gamma$.
When α_g is not inner for any $g \in \Gamma \setminus \{1\}$,
(α, u) is said to be outer.
When $u \equiv 1$, $\alpha : \Gamma \curvearrowright A$ is a genuine action.

Cocycle conjugacy

Definition

Two cocycle actions $(\alpha, u) : \Gamma \curvearrowright A$ and $(\beta, v) : \Gamma \curvearrowright B$ are said to be cocycle conjugate if there exist a family of unitaries $(w_g)_{g\in\Gamma}$ in B and an isomorphism $\theta : A \to B$ such that

$$\theta \circ \alpha_g \circ \theta^{-1} = \operatorname{Ad} w_g \circ \beta_g$$

and

$$\theta(u(g,h)) = w_g \beta_g(w_h) v(g,h) w_{gh}^*$$

hold for every $g, h \in \Gamma$.

Our eventual goal is

- to classify the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$,
- to classify (α, u) up to cocycle conjugacy and to determine when (α, u) is cocycle conjugate to a genuine action.

Twisted crossed product

Definition

For $(\alpha, u) : \Gamma \curvearrowright A$, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is the universal C^*-algebra generated by A and a family of unitaries $(\lambda_g^\alpha)_{g\in\Gamma}$ satisfying

$$\lambda_g^\alpha \lambda_h^\alpha = u(g,h)\lambda_{gh}^\alpha \quad \text{and} \quad \lambda_g^\alpha a \lambda_g^{\alpha *} = \alpha_g(a)$$

for all $g, h \in \Gamma$ and $a \in A$.

If (α, u) and (β, v) are cocycle conjugate via $\theta : A \to B$ and $(w_g)_g$, then $A \rtimes_{(\alpha,u)} \Gamma$ and $B \rtimes_{(\beta,v)} \Gamma$ are canonically isomorphic by

$$a \mapsto \theta(a) \quad \text{and} \quad \lambda_g^\alpha \mapsto w_g \lambda_g^\beta.$$

Group actions on injective factors

Theorem

Let M be an injective factor. Two cocycle actions (α, u) and (β, v) of Γ on M are strongly cocycle conjugate if and only if $\mathrm{Inv}(\alpha, u) = \mathrm{Inv}(\beta, v)$.

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The invariant $\mathrm{Inv}(\alpha, u)$ consists of "centrally trivial part", "Connes-Takesaki module" and "characteristic invariant".

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The invariant $\mathrm{Inv}(\alpha, u)$ consists of "centrally trivial part", "Connes-Takesaki module" and "characteristic invariant".
The theorem above has a long history: A. Connes (cyclic groups on II_1), V. F. R. Jones (finite groups on II_1), A. Ocneanu (on II_1), C. E. Sutherland and M. Takesaki (on III_λ with $\lambda \neq 1$), Y. Kawahigashi, C. E. Sutherland and M. Takesaki (abelian groups on III_1), Y. Katayama, C. E. Sutherland and M. Takesaki (all actions)...
and T. Masuda (all cocycle actions with a shorter proof).

$\mathcal{Z}$-stability of crossed product

Theorem (Y. Sato and M)

Let A be a stably finite C^*-algebra with finite nuclear dimension and with finitely many extremal tracial states. Let Γ be an elementary amenable group.
Let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action.
Then (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1) : \Gamma \curvearrowright A \otimes \mathcal{Z}$.
In particular, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is $\mathcal{Z}$-stable.

In order to prove this, it suffices to construct a unital embedding of $\mathcal{Z}$ into the fixed point algebra $(A^\infty \cap A')^\alpha$.

Strong outerness

Let $T(A)$ denote the set of tracial states and let π_τ be the GNS representation by $\tau \in T(A)$.

Definition

$\alpha \in \mathrm{Aut}(A)$ is said to be not weakly inner if the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is not inner for any $\tau \in T(A)^\alpha$, that is, there does not exist a unitary $U \in \pi_\tau(A)''$ such that $\bar{\alpha} = \mathrm{Ad}\, U$.
A cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ is said to be strongly outer if α_g is not weakly inner for every $g \in \Gamma \setminus \{1\}$.

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If $T(A) = \{\tau\}$, then

$$(\alpha, u) : \Gamma \curvearrowright A \text{ is strongly outer} \iff (\bar{\alpha}, u) : \Gamma \curvearrowright \pi_\tau(A)'' \text{ is outer.}$$

Elementary amenable groups

Definition

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For instance, all solvable groups are elementary amenable.
There exist amenable groups which are not elementary (R. I. Grigorchuk).

Weak Rohlin property

Theorem

Let A be a nuclear stably finite C^*-algebra with finitely many extremal tracial states and let Γ be elementary.
Then any strongly outer cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ has the weak Rohlin property, i.e.

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Theorem

Let A be a nuclear stably finite C^*-algebra with finitely many extremal tracial states and let Γ be elementary.
Then any strongly outer cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ has the weak Rohlin property, i.e. for any $F \Subset \Gamma$ and $\varepsilon > 0$, there exist an (F, ε)-invariant $K \Subset \Gamma$ and a sequence $(e_n)_n$ of positive contractions in A such that

$$[e_n, a] \to 0, \quad \alpha_g(e_n)\alpha_h(e_n) \to 0, \quad \tau(e_n) \to |K|^{-1}$$

as $n \to \infty$ for all $a \in A$, $g, h \in K$ with $g \neq h$ and $\tau \in T(A)$.

A bounded sequence $(x_n)_n$ in A is called a central sequence if $[x_n, a] \to 0$ as $n \to \infty$ for all $a \in A$.

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Let A be a stably finite C^*-algebra with finite nuclear dimension. Then A has the property (SI), i.e. for any central sequences $(x_n)_n$ and $(y_n)_n$ in A satisfying $0 \leq x_n, y_n \leq 1$,

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \tau(x_n) = 0 \quad \textit{and} \quad \inf_{m\in\mathbb{N}} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(y_n^m) > 0,$$

there exists a central sequence $(s_n)_n$ in A such that

$$\lim_{n\to\infty} \|s_n^* s_n - x_n\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|y_n s_n - s_n\| = 0.$$

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By using the weak Rohlin property and the property (SI), we can construct a unital embedding of $\mathcal{Z}$ into the fixed point algebra $(A^\infty \cap A')^\alpha$, which implies the $\mathcal{Z}$-absorption theorem.

$\mathbb{Z}$-actions on UHF algebras

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property, i.e.

$\mathbb{Z}$-actions on UHF algebras

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property, i.e. for any $m \in \mathbb{N}$, there exist central sequences of projections $(e_n)_n, (f_n)_n$ in A such that

$$\sum_{i=0}^{m-1} \alpha^i(e_n) + \sum_{j=0}^{m} \alpha^j(f_n) \to 1.$$

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Theorem (A. Kishimoto 1995)

Let A be a UHF algebra. All strongly outer $\mathbb{Z}$-actions on A are cocycle conjugate to each other.

The proof uses the Evans-Kishimoto intertwining argument, which is an equivariant version of Elliott's intertwining argument.

Stability

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We are given $\alpha, \beta : \Gamma \curvearrowright A$. Suppose that there exists a family $(u_g)_{g\in\Gamma}$ of unitaries in A^∞ such that

$$u_g\alpha_g(u_h) = u_{gh},\ \ \beta_g(a) = (\mathrm{Ad}\, u_g \circ \alpha_g)(a) \quad \forall a \in A,\ \ g, h \in \Gamma.$$

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Stability of α implies there exists a unitary $v \in A^\infty$ such that

$$u_g = v\alpha_g(v^*) \quad \forall g \in \Gamma.$$

Then we would have

$$\beta_g(a) = (\mathrm{Ad}\, v \circ \alpha_g \circ \mathrm{Ad}\, v^*)(a) \quad \forall a \in A,\ g \in \Gamma,$$

which may induce ‘conjugacy’ between α and β.

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Theorem (M 2010)

Let A be a unital simple AH algebra with slow dimension growth, real rank zero and finitely many extremal tracial states. Let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. If α_k is approximately inner for some $k \in \mathbb{N}$, then α has the Rohlin property.

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Theorem (M 2010)

Let A be a unital simple AH algebra with slow dimension growth and real rank zero. If two actions $\alpha, \beta : \mathbb{Z} \curvearrowright A$ have the Rohlin property and $\alpha_1 \circ \beta_{-1}$ is asymptotically inner, then α and β are cocycle conjugate.

$\mathbb{Z}$-actions on $\mathcal{Z}$

Theorem (Y. Sato 2010)

All strongly outer $\mathbb{Z}$-actions on $\mathcal{Z}$ are cocycle conjugate to each other.

We sketch the proof. Let $\alpha, \beta : \mathbb{Z} \curvearrowright \mathcal{Z}$ be strongly outer.

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We sketch the proof. Let $\alpha, \beta : \mathbb{Z} \curvearrowright \mathcal{Z}$ be strongly outer.
(1) By the theorem mentioned before, we may replace α, β with $\alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id} : \mathbb{Z} \curvearrowright \mathcal{Z} \otimes \mathcal{Z}$.

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(3) By Kishimoto's result, $\alpha \otimes \mathrm{id}$ and $\beta \otimes \mathrm{id}$ are cocycle conjugate as actions on $\mathcal{Z} \otimes B$ with B being a UHF algebra.

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(4) With some extra effort we get cocycle conjugacy on $\mathcal{Z} \otimes \mathcal{Z}$.

Cocycle actions of $\mathbb{Z}^2$ on AF algebras (1/2)

We write $\mathbb{Z}^2 = \langle a, b \mid bab^{-1} = a \rangle$.

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Theorem (H. Nakamura 1999, M 2010, Y. Sato and M)

Let A be a unital simple AF algebra with finitely many extremal tracial states and let $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$ be a strongly outer cocycle action. Suppose that α_a^n and α_b^n are approximately inner for some $n \in \mathbb{N}$. Then (α, u) has the Rohlin property.

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For a cocycle action $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$, we have

$$\begin{aligned}\alpha_b \circ \alpha_a &= \mathrm{Ad}\, u(b, a) \circ \alpha_{ba} \\ &= \mathrm{Ad}\, u(b, a) \circ \alpha_{ab} = \mathrm{Ad}(u(b, a)u(a, b)^*) \circ \alpha_a \circ \alpha_b\end{aligned}$$

Conversely, two single automorphisms commuting up to an inner automorphism give rise to a cocycle action of $\mathbb{Z}^2$.

Cocycle actions of $\mathbb{Z}^2$ on AF algebras (2/2)

For $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$ satisfying $\alpha_g \in \overline{\mathrm{Inn}}(A)$ $\forall g \in \mathbb{Z}^2$, we introduce an invariant $c(\alpha, u) \in \mathrm{OrderExt}(K_0(A), K_0(A))$ as follows:

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Let $\tilde{\eta}_0 : \overline{\mathrm{Inn}}(B) \to \mathrm{OrderExt}(K_1(B), K_0(B))$ be the homomorphism introduced by A. Kishimoto and A. Kumjian.
Define $c(\alpha, u) = \tilde{\eta}_0(\tilde{\alpha}_b) \in \mathrm{OrderExt}(K_1(B), K_0(B))$, which can be identified with $\mathrm{OrderExt}(K_0(A), K_0(A))$.

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Let $\tilde{\eta}_0 : \overline{\mathrm{Inn}}(B) \to \mathrm{OrderExt}(K_1(B), K_0(B))$ be the homomorphism introduced by A. Kishimoto and A. Kumjian.
Define $c(\alpha, u) = \tilde{\eta}_0(\tilde{\alpha}_b) \in \mathrm{OrderExt}(K_1(B), K_0(B))$, which can be identified with $\mathrm{OrderExt}(K_0(A), K_0(A))$.

Theorem (M 2010, Y. Sato and M)

Let A be as before. Let (α, u) and (β, v) be strongly outer cocycle actions of $\mathbb{Z}^2$ such that $\alpha_g, \beta_g \in \overline{\mathrm{Inn}}(A)$.
If $c(\alpha, u) = c(\beta, v)$, then (α, u) and (β, v) are cocycle conjugate.

Cocycle actions of $\mathbb{Z}^2$ on UHF algebras

When A is a UHF algebra, we have

$$\mathrm{OrderExt}(K_0(A), K_0(A)) \cong \mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A)).$$

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Corollary (T. Katsura and M 2008, Y. Sato and M)

Let A be a UHF algebra. There exists a natural bijective correspondence between the following two sets.

1. *Cocycle conjugacy classes of strongly outer cocycle actions of $\mathbb{Z}^2$ on A.*
2. $\mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A))$.

Moreover, genuine actions correspond to

$$\{r \in \mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A)) \mid r([1]) = 0\}.$$

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Let $(\alpha, u) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be a cocycle action.
As before, put $\check{u} = u(b, a)u(a, b)^*$.
The following theorem says that the de la Harpe-Skandalis determinant $\Delta_\tau(\check{u}) \in \mathbb{R}/\mathbb{Z}$ is the complete invariant of (α, u).

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Theorem (Y. Sato and M)

Let $(\alpha, u), (\beta, v) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be strongly outer cocycle actions. Then they are cocycle conjugate if and only if $\Delta_\tau(\check{u}) = \Delta_\tau(\check{v})$.

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Theorem (Y. Sato and M)

Let $(\alpha, u), (\beta, v) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be strongly outer cocycle actions. Then they are cocycle conjugate if and only if $\Delta_\tau(\check{u}) = \Delta_\tau(\check{v})$.

The proof uses the same idea as $\mathbb{Z}$-actions:

- (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1)$ on $\mathcal{Z} \otimes \mathcal{Z}$.
- We have already classified $(\alpha \otimes \mathrm{id}, u \otimes 1)$ on $\mathcal{Z} \otimes B$ with B being a UHF algebra.
- Some extra effort gives the conclusion.

$\mathbb{Z}^N$-actions on UHF algebras of infinite type

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Theorem (M)

Let A be a UHF algebra of infinite type. Then, all strongly outer actions of $\mathbb{Z}^N$ on A are mutually cocycle conjugate to each other.

Actions of the Klein bottle group (1/2)

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Let A be a UHF algebra and let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action. Then for any $m \in \mathbb{N}$, there exist central sequences of projections $(e_n)_n, (f_n)_n$ in A such that

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All strongly outer cocycle actions of Γ on a UHF algebra are mutually cocycle conjugate.

Actions of the Klein bottle group (2/2)

We sketch the proof.
Letting $\check{u} = u(b,a)u(a^{-1},b)^*$, we have $\alpha_b \circ \alpha_a = \mathrm{Ad}\,\check{u} \circ \alpha_a \circ \alpha_b$, and α_b extends to $\tilde{\alpha}_b \in \mathrm{Aut}(A \rtimes_{\alpha_a} \mathbb{Z})$ by $\tilde{\alpha}_b(\lambda^{\alpha_a}) = \check{u}\lambda^{\alpha_a *}$.

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asymptotically inner. Then the Evans-Kishimoto intertwining argument works and yields the conclusion.

Open problems

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- Classify strongly outer $\mathbb{Z}$-actions on ‘classifiable’ C^*-algebras (as much as possible).

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